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A Method from Categories for Introducing a General Notion of Convergence and Limit

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An "axiomatic" construction, obtained by means of Kan extensions of suitable patterns of functors, is proposed, leading to a unified concept which covers in a natural nontrival way many classical topics of mathematical analysis (e.g., the concepts of limit, of semicontinuous envelope, the various notions of convergence, including some more recent ideas).

1. INTRODUCTION

This paper does not contain, properly, any "new theorem." Rather, it wants to propose an abstract construction, essentially of algebraic nature, which provides a sort of "axiomatization" of some classical topics of Mathematical Analysis, as well as possibly a convenient frame for understanding and generalizing some "global" properties of them.

This general construction is obtained by using a typical technique taken from Category theory: it is mainly given by (repeated) Kan extensions [6, 8] of suitable patterns of functors, as we shall describe in the next section. In the last section, we will test this scheme, by reexamining some key concepts of analysis, such as the notion of limit, of semicontinuous envelope, the various notions of convergence, including some more recent and refined ideas.

2. BASIC CONSTRUCTIONS

In the following, P will denote a complete partially ordered set; $a, a', \ldots$, will be the elements of P and $a \succ a'$ the order relation; X any space and finally P^X the set of all functions $f\colon X \to P$. The set P^X also will be equipped with a (pre-) order relation denoted $\gtrsim$ (the symbol $\geqslant$ will be reserved only for the standard ordering of real numbers). Finally, let $J\colon P \to P^X$ denote the map sending each $a \in P$ to the constant function $f(x) = a$ for all $x \in X$. It is understood that all the (pre-)ordered or partially ordered sets we will deal with, will

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be always considered as categories, in the usual way, identifying arrows with order relations, namely, $a \to a'$ when $a \succ a'$, and $f \to f'$ when $f \gtrsim f'$.

The following three definitions contain all the basic material needed for our purposes. Some other results, centered around similar ideas, can be found in [3, 4, 7] and references therein.

DEFINITION 1. Let the preorder relation $\gtrsim$ defined in P^X be such that $a \succ a'$ (in P) implies $Ja \gtrsim Ja'$ (in P^X); then the map J can be viewed as a (trivial) functor. Define now K^+ (resp. K^-) the left (resp. right) Kan extension $K^\pm\colon P^X \to P$ of the identity $I\colon P \to P$ along $J\colon P \to P^X$. A useful variant can be the following: if P is not complete, and $\bar{P}$ is its completion, the functors $K^\pm$ can be defined as the extensions $K^\pm\colon P^X \to \bar{P}$ of the inclusion $\bar{I}\colon P \to \bar{P}$ along $J\colon P \to P^X$.

a. *Simple Examples.* (i) If $P = \bar{R}$, the extended real line with the standard ordering $\geqslant$, X is any space and $f \gtrsim f'$ is assumed to mean $f(x) \geqslant f'(x)$ for all $x \in X$, then $K^+f = \sup_{x\in X} f(x)$, and $K^-f = \inf_{x\in X} f(x)$.

(ii) More in general, if P is any complete, partially ordered set, X can be viewed as an arbitrary set of indices j, and $\gtrsim$ means that $a_j \succ a_j'$ for all j, then $K^+\{a_j\}$ = least upper bound of $\{a_j\}$ and K^- = greatest lower bound.

(iii) If $P = \bar{R}$, X the set N of integers, and P^X (i.e., the real sequences $\{a_n\}$) is ordered by the rule: $\{a_n\} \gtrsim \{a_n'\}$ if $a_n \geqslant a_n'$ except for at most a finite number of indices, then $K^+\{a_n\} = \limsup a_n$, and $K^- = \liminf$.

DEFINITION 2 (Iterated extensions). Let X_1, X_2 be two spaces, and $Y_1 : P^{X_1} \to P^{X_1\times X_2}$ the map sending each function $f_1(x_1)$ of the "first variable" x_1 to the function $f\colon (x_1, x_2) \to f_1(x_1)$. If in $P^{X_1\times X_2}$ a preordering is given such that $f_1 \gtrsim f_1'$ in P^{X_1} implies $Y_1 f_1 \gtrsim Y_1 f_1'$ in $P^{X_1\times X_2}$, then Y_1 is a functor (see below the scheme of functors), and one can introduce the left and the right

$$\begin{tikzcd}
 & & P^{X_1\times X_2} \arrow[dd, "K^{\pm\pm}"] \\
 & P^{X_1} \arrow[ur, "Y_1"] \arrow[dr, "K_1^{\pm}"] & \\
P \arrow[ur, "J"] \arrow[rr, "I"'] & & P
\end{tikzcd}$$

Kan extension K^{++} and K^{+-} of the functor K^+ (here called $K_1{}^+$) introduced in Definition 1 and, respectively, the left and right extensions K^{-+} and K^{--} of K^- (here $K_1{}^-$) of Definition 1 along Y_1.

In all the remaining part of this paper, the space X will be assumed to be a topological space. We specialize now the preordering in P^X according to the following:

DEFINITION 3. Let ξ be a fixed point in X. Define $f \gtrsim f'$ in P^X if there is a neighbourhood U_ξ of ξ such that $f(x) \succ f'(x)$ for all $x \in U_\xi$. Analogously, if $X = X_1 \times X_2$, where X_1 and X_2 are two topological spaces with their respective topologies, and $\xi \equiv (\xi_1, \xi_2)$ a point in $X_1 \times X_2$, we define $f \gtrsim f'$ in $P^{X_1 \times X_2}$ if there are U_1 and U_2 such that $f(x_1, x_2) \succ f'(x_1, x_2)$ for all $x_1 \in U_1$ and $x_2 \in U_2$. Then, through Definitions 1 and 2, one comes equipped with the Kan extension functors, which will be denoted now by $K_\xi^{\pm}$, K_ξ^{++}, etc.

In order to clarify both the unifying character of this scheme, and its possible applications, we will devote the next section to reconsider some classical, and even elementary, cases. Let us briefly note here a technical point. One can observe that in Definition 2 the role of the spaces X_1 and X_2 is "asymmetrical": in fact, if the role of X_1 and X_2 is exchanged (once given the function $f(x_1, x_2)$, consider first the functions $f_2(x_2)$ and the extensions $K_2^{\pm}: P^{X_2} \to P$, etc.), and if the four functors so obtained are denoted by $\tilde{K}^{\pm\pm}$, one can verify that the following relations hold, for any f,

$$K^{++}f = \tilde{K}^{++}f; \qquad K^{--}f = \tilde{K}^{--}f;$$
$$K^{-+}f \succ \tilde{K}^{+-}f; \qquad \tilde{K}^{-+}f \succ K^{+-}f.$$

3. Particular Cases and Applications

I. *Real Functions*

In this subsection, we take $P = \bar{R}$.

a. *Limits of Real Functions.* If $\xi \in X$, the functors K_ξ^+ and K_ξ^- of Definition 3 give the upper limit (i.e., the "lowest" of the real numbers c such that $c \gtrsim f$) and the lower limit of the function $f \in P^X$ when $x \to \xi$ [3]. A similar interpretation, in the product topology, if $\xi \equiv (\xi_1, \xi_2)$, holds for K_ξ^{++} and K_ξ^{--}, whereas K_ξ^{+-} and K_ξ^{-+} coincide just with the "multiple Γ-limits of f for $(x_1, x_2) \to (\xi_1, \xi_2)$," as defined in [5].

b. *Semicontinuous Envelopes.* If in Definition 3 one fixes the function f and variates the point ξ in X, the new functions of ξ so obtained are just the semicontinuous envelopes f^*, f_* of the real function f. But this procedure can be repeated also in the abstract case of Definition 3: the functions defined by $f^{\pm}(\xi) = K_\xi^{\pm}(f)$ can be viewed as "generalized envelopes" of f. In fact, one can show, e.g., that $[f^+]^+ = f^+$.

c. *Sequences of functions. Γ-limits.* Let us take in Definition 2, $X_1 = N$ and $X_2 = X$: then one is considering sequences $\{f_n(x)\}$ of real functions ($n = 1, 2, \ldots$; $x \in X$). For fixed $\xi \in X$, let us put $\{f_n\} \gtrsim \{f'_n\}$ if there are a neighbourhood U_ξ and a number ν such that $f_n(x) \geqslant f'_n(x)$ for any $x \in U_\xi$ and $n \geqslant \nu$.[1] Finally, if $K_1^{\pm}$ are the functors "lim sup" and "lim inf" of Example iii in Section 2, then one obtains through Definition 3 the four "limits" $K^{\pm\pm}_{\infty,\xi}$. Their meaning is clear: to give an example, one has

$$K^{+-}_{\infty,\xi}\{f_n\} = c$$

if and only if "*for any* $\epsilon > 0$ *and* $\epsilon' > 0$, $\exists\, \bar{\nu}$ *such that*, $\forall U_\xi$, $\exists y \in U_\xi$ *satisfying*

$$c + \epsilon \geqslant f_n(y), \quad \forall n \geqslant \bar{\nu}$$

and simultaneously $\exists \bar{U}_\xi$ *such that*, $\forall \nu$, $\exists m \geqslant \nu$ *satisfying*

$$c - \epsilon' \leqslant f_m(x), \qquad \forall x \in \bar{U}_\xi."$$

Recently, another notion of convergence has been usefully introduced in the literature (see, e.g., [2] and references therein). This can be obtained from our abstract scheme by simply changing the preordering in $P^{N \times X}$ according to this rule: fixed $\xi \in X$, we say $\{f_n\} \gtrsim \{f'_n\}$ if, for any sequence of points $\{x_n\}$ tending to ξ, one has $f_n(x_n) \geqslant f'_n(x_n)$ except for at most a finite number of indices. If $H_\xi^{\pm\pm}$ denote the four functors obtained in this way, when, e.g., $H_\xi^{+-}\{f_n\} = H_\xi^{--}\{f_n\}$ for a given sequence $\{f_n(x)\}$, then this sequence is said to "(sequentially) Γ^--converge for $x \to \xi$."

II. *Set theory*; *Maps and Sequences*

Choose now $P = \{0, 1\}$, the set of the two numbers 0 and 1; then $P^X \simeq \mathscr{P}(X)$, the class of all subsets of X, and each subset S turns out to be specified by its characteristic function $\chi_S(x)$, whereas $J(0) = \varnothing$ and $J(1) = X$. For a fixed point $\xi \in X$, the preordering given in Definition 3 becomes now equivalent to the following rule: we say $S \gtrsim S'$ if there is a neighbourhood U_ξ such that $U_\xi \cap S \supset U_\xi \cap S'$.

a. *Sets.* By means of Definition 3 and 1, we obtain the functors $K_\xi^{\pm}: \mathscr{P}(X) \to P$ defined on each subset S of X, with values either 0 or 1. It is easily seen that $K_\xi^{+}S = 1$ ($K_\xi^{-}S = 1$) if and only if ξ is in the closure (the internal part) of S. Through Definition 3 and 2, one extends the construction to product spaces $X = X_1 \times X_2$: the set of all points $\xi \equiv (\xi_1, \xi_2)$ such that, say, $K_\xi^{+-}S = 1$ is precisely what in Ref. [5] is defined to be the "G^{+-}-limit of the set S."

[1] This is precisely the preorder which could be deduced from Definition 2 if one defines, as usual, a neighborhood of the infinity as the set of the integers larger than some ν.

b. *Maps.* Consider a product space $X_1 \times X_2$ and let S be a subset with this property: for each $x_1 \in X_1$, there is exactly one $x_2 \in X_2$ such that $(x_1, x_2) \in S$, (equivalently, a one-valued map $f: X_1 \to X_2$ is defined). Note first that if for a point $\xi \equiv (\xi_1, \xi_2)$ one has $K_\xi^{-+}S = 1$, then also $K_\xi^{++}S = 1$, and that in general (unless X_2 is equipped with a discrete topology) $K_\xi^{\pm -}S = 0$ for any ξ and S. The meaning of $K_\xi^{\pm +}$ easily follows from their definition. For example,

$$K_\xi^{++}S = 1$$

when "*for any neighbourhood* U_2 *of* ξ_2, *there are points* y_1 *arbitrarily near to* ξ_1 *(in the topology of* X_1*) such that* $f(y_1) \in U_2$."

Similarly, $K_\xi^{-+}S = 1$ when ξ_2 is the *limit* of f for x_1 tending to ξ_1 (with respect to the given topologies). It can happen that for a given $\xi_1 \in X_1$ there is only one ξ_2 such that $K_\xi^{++}S = 1$ but $K_\xi^{-+}S = 0$ with $\xi \equiv (\xi_1, \xi_2)$; this is the case, e.g., when $X_1 = X_2$ is a Hilbert space with its norm topology, f is a *closed unbounded* operator and ξ_1 a vector in the domain of this operator.

c. *Sequences of Maps.* The case of sequences of maps $f_n : X_1 \to X_2$ can be easily cast in this scheme. Consider in the class $P^{N \times X_1 \times X_2}$, i.e., the class of sequences $\{S_n\}$ of the subsets of $X_1 \times X_2$, a preorder similar to that in the above cases, namely, for a fixed point $\xi \equiv (\xi_1, \xi_2)$, we say $\{S_n\} \gtrsim \{S'_n\}$ if there are neighbourhoods U_1 of ξ_1 and U_2 of ξ_2 and a number ν such that $U \cap S_n \supset U \cap S'_n$, where $U = U_1 \times U_2$, for any $n \geqslant \nu$. Each map f_n of the sequence $\{f_n\}$ can be identified with a set S_n having the property stated at the beginning of b above. Finally, a further Kan extension is needed: the chain of inclusions is $P \to P^N \to P^{N \times X_1} \to P^{N \times X_1 \times X_2}$, and for any ξ one obtains four relevant functors $K_{\infty,\xi}^{\pm\pm +}$ (in general, $K_{\infty,\xi}^{\pm\pm -}\{f_n\} = 0$ for any ξ, as in b). Their meaning is easily deduced from the above discussion; for instance,

$$K_{\infty,\xi}^{--+}\{f_n\} = 1 \qquad \text{with} \quad \xi = (\xi_1, \xi_2)$$

means that:

$$\text{“}\forall U_2, \qquad \exists \nu \qquad \textit{and} \qquad \exists U_1$$

such that

$$f_n(x_1) \in U_2, \qquad \forall n \geqslant \nu, \qquad \forall x_1 \in U_1 \text{,”}$$

where U_1 stands for a neighbourhood of ξ_1 in the topology of X_1, and similar for U_2. The next subsection will clarify some interesting relations existing between the functors $K^{\pm\pm +}$ and some properties of various types of convergence for sequences of linear operators.

III. *Sequences of Operators. G-Convergence*

This paragraph is a concrete realization of the above case IIc. In fact, X_1 and X_2 will be assumed here to be a Hilbert space H, and f_n linear bounded operators $A_n \in \mathscr{L}(H)$. We shall write $K_\xi^{\pm\pm+} A_n$ instead of $K_{\infty,\xi}^{\pm\pm+}\{f_n\}$. Some interesting situations (see also [1]) follow.

a. *Strong Convergence.* Let $X_1 = H = X_2$ be equipped with the norm topology. Then the following result holds: If A_n converges strongly to an operator A_0, then one has $K_\xi^{--+} A_n = 1$ (and thus a fortiori $K_\xi^{-++} A_n = 1$) if and only if ξ belongs to the graph of A_0. Viceversa, given a sequence of operators A_n, if for each vector $\xi_1 \in X_1$ there is exactly one ξ_2 such that $K_\xi^{-++} A_n = 1$ with $\xi = (\xi_1, \xi_2)$, then the operator A_0 defined by $A_0(\xi_1) = \xi_2$ is the strong limit of A_n.

b. *Weak Convergence.* Let now $X_1 = H$ be equipped with the norm topology and $X_2 = H$ with the weak topology. Then, the weak convergence of A_n to an operator A_0 implies, with the same notations as in a, $K_\xi^{--+} A_n = K_\xi^{-++} A_n = 1$; vice versa, by finding as above ξ such that $K_\xi^{-++} A_n = 1$, an operator turns out to be defined, which is the weak limit of A_n.

c. *G-Convergence.* Suppose that $X_1 = H$ is equipped with the weak topology whereas $X_2 = H$ with the norm topology. In this case one has the following slightly different result:

Let $\{A_n\}$ be a sequence of invertible operators such that $\{A_n^{-1}\}$ is *weakly* convergent to an invertible operator $B \in \mathscr{L}(H)$, this implies $K_\xi^{-++} A_n = 1$, where $\xi = (\xi_1, \xi_2)$ with $\xi_2 = B^{-1}(\xi_1)$. Vice versa, given a sequence of invertible A_n, if one finds $K_\xi^{-++} A_n = 1$ for pairs $\xi = (\xi_1, \xi_2)$ in such a way that this realizes a bijective correspondence between the vectors ξ_1 and ξ_2, then the operator defined by $B(\xi_2) = \xi_1$ is the weak limit of A_n^{-1}.[2]

Proof. We outline the proof of c (for a and b the proof is similar). Observe first that $K_\xi^{--+} A_n = 1$ (resp. $K_\xi^{-++} A_n = 1$) with $\xi = (\xi_1, \xi_2)$ means now that for any strong neighbourhood $U_2{}^s$ of ξ_2 there are a number ν and a weak neighbourhood $U_1{}^w$ of ξ_1 such that $A_n(x_1) \in U_2{}^s$ for all $n \geqslant \nu$ and all $x_1 \in U_1{}^w$ (resp. for any $U_2{}^s$, there are ν and vectors y_1 belonging to arbitrarily small $U_1{}^w$ such that $A_n(y_1) \in U_2{}^s$ for all $n \geqslant \nu$). Then clearly the hypothesis $A_n^{-1} \to^w B$ does *not* imply $K_\xi^{--+} A_n = 1$ with $\xi_2 = B^{-1}(\xi_1)$ (and this distinguishes c from a and b). It is true, instead, that taking the vectors x_2 of any strong $U_2{}^s$, the vectors $A_n^{-1}(x_2)$ are arbitrarily near, in the weak sense, to ξ_1 for any sufficiently large n, as shown by this inequality

$$\begin{aligned} |(v, A_n^{-1}(x_2) - \xi_1)| &\leqslant |(v, A_n^{-1}(x_2) - A_n^{-1}(\xi_2))| + |(v, A_n^{-1}(\xi_2) - \xi_1)| \\ &\leqslant \| v \| \| A_n^{-1} \| \| x_2 - \xi_2 \| + |(v, A_n^{-1}(\xi_2) - B(\xi_2))| \end{aligned}$$

for arbitrary $v \in H$, and then just $K_\xi^{-++} A_n = 1$. Vice versa, given a sequence of invertible A_n, if for each ξ_1 one bijectively finds a ξ_2 such that $K_\xi^{-++} A_n = 1$,

[2] Note that in general B^{-1} does not coincide with the weak limit of A_n.

then one has, for any $\epsilon > 0$, $\| A_n(y_1) - \xi_2 \| < \epsilon$ for vectors y_1 arbitrarily near (in the weak sense) to ξ_1, and, defining $B(\xi_2) = \xi_1$,

$$\begin{aligned} |(v, A_n^{-1}(\xi_2) - B(\xi_2))| &\leqslant |(v, A_n^{-1}(\xi_2) - y_1)| + |(v, y_1 - B(\xi_2))| \\ &\leqslant \| v \| \| A_n^{-1} \| \| \xi_2 - A_n(y_1) \| + |(v, y_1 - \xi_1)| \end{aligned}$$

which shows that B is the weak limit of A_n^{-1}.

Many other applications of this scheme are easily conceivable. For instance, the above cases are easily generalized taking for X_1 a Banach space and for X_2 its dual space. One can also modify the preordering in $P^{N \times X_1 \times X_2}$ in a similar way as for introducing the sequential Γ-limits (Ic) (See [1]). In this situation, IIIc gives precisely a definition of the *G-convergence* of operators, a notion which has revealed a very useful tool in many problems (see, e.g., [5, 9] and the references therein).

Note added in proof. The precise categorical characterization of various types of convergence exposed above (in III) for linear operators, can actually be proved to hold—essentially unchanged—for sequences of arbitrary functions f_n. See: G. Cicogna, Un'analisi categoriale delle nozioni di convergenza, *Boll. Un. Mat. Ital.*, in press.

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